Zeszyty Naukowe WWSI, No. 11, Vol. 8, 2014, pp. 7-27

Implementation of a Value for Generalized Characteristic Function Games

Tomasz P. Michalak*[1,2] and Talal Rahwan[3,4]

[1]Department of Computer Science, University of Oxford, UK
[2]Institute of Informatics, University of Warsaw, Poland
[3]Masdar Institute of Science and Technology, UAE
[4]Warsaw School of Computer Science, Poland

Abstract

Generalized characteristic function games are a variation of characteristic function games, in which the value of a coalition depends not only on the identities of its members, but also on the order in which the coalition is formed. This class of games is a useful abstraction for a number of realistic settings and economic situations, such as modeling relationships in social networks. To date, two main extensions of the Shapley value have been proposed for generalized characteristic function games: the Nowak-Radzik value and the Sánchez-Bergantiños value. In this context, the present article studies generalized characteristic function games from the point of view of implementation and computation. Specifically, the article presents a non-cooperative mechanism that implements the Nowak-Radzik value in Subgame-Perfect Nash Equilibria in expectation.

Keywords – generalized characteristic function games, Shapley value implementation

1 Introduction

Coalitional games are an important model for many realistic economic situations that capture the ability of players to take joint, coordinated actions. Typically, a coalitional game model specifies payoffs attainable by various subsets (or coalitions) of players cooperating within the game. Given these payoffs, fundamental game-theoretic research questions concerning coalitional games include: (i) Which coalition will actually form? (ii) How should the coalitional payoff be distributed among coalition members? Moreover, assuming that desirable coalitions and payoff distribution methods exist: (iii) How can we create a mechanism that implements a specific solution in an environment of self-interested players? Coalitional games also raise

*Email: tomasz.michalak@cs.ox.ac.uk

many important questions from the computer-science perspective, key among them being: (iv) How to represent games compactly? and (v) How to efficiently compute their solutions given such compact representations [1]. This article addresses the third question for coalitional games in the *generalized characteristic function form* as introduced by Nowak and Radzik [2]. This class of games generalizes characteristic function games with transferable utility by distinguishing between different orders in which players create coalitions. Thus, in this model, the value of a coalition depends not only on its members, but also *on the order in which those members joined the coalition*.

Generalized characteristic function form games naturally capture a number of real-world situations. Consider, for example, a search on a social network where we need to answer a question that only a few nodes can answer, and the question is propagated through referrals along the connections of each node. This was the case, for instance, with the recent *TAG challenge* [3, 4], where photos of five suspects were announced on a particular date, along with the name of the city where each criminal was located, and the challenge was to take photos of as many suspects as possible within 12 hours using referrals on social networks. In such cases, the order in which nodes are added to the search influences the time required to find an answer (e.g., the sooner the nodes with more connections join, the faster the search becomes). As a second example, consider the cost-allocation problem studied by Sánchez and Bergantiños [5], where a group of universities participating in a joint research project invite a foreign expert for a visit. The budget of such a visit will depend on the planned route, i.e., the sequence in which the universities are to host the researcher.

Clearly, situations such as the above cannot be captured within a conventional coalitional game model (i.e., a game in characteristic function form), where the value of a coalition depends solely on the identity of its members, without considering the order in which the members have joined it. Consequently, a growing body of work has considered generalized characteristic function games. In this context, a number of researchers have focused on the issue of fair payoff division. The most well-known fair payoff division concept in coalitional games is the Shapley value [6]. The basic idea is that a_i's payoff should be equal to a_i's average marginal contribution, taken over all possible ways in which players could join the game (and contribute to the creation of each coalition's value). For instance, in the game of three players, there are altogether 3! ways in which players could join the game: (i) a_1, a_2, a_3, (ii) a_1, a_3, a_2, (iii) a_2, a_1, a_3, (iv) a_3, a_1, a_2, (v) a_2, a_3, a_1, and (vi) a_3, a_2, a_1. As such, there are 3! corresponding marginal contributions of a_1: in (i) and (ii) a_1 joins first (i.e., a_1 *contributes twice to the empty set*); in (iii) a_1 joins after a_2 (i.e., a_1 *contributes once to* $\{a_2\}$); in (iv) a_1 joins after a_3 (i.e., a_1 *contributes once to* $\{a_3\}$); finally in (v) and (vi) a_1 joins last (i.e., a_1 *contribute twice to* $\{a_2, a_3\}$). The average of all these marginal contributions is the Shapley value of a_1.

There are two main extensions of the Shapley value to generalized characteristic function games. The first was proposed by Nowak and Radzik [2] (which we refer to as the *NR value*), while the second was introduced by Sánchez and Bergantiños [7] (which we refer to as the *SB value*). The difference between these values can be seen in cases (v) and (vi) of the above 3-agent example. In particular, if v denotes the characteristic function, and v_g denotes the *generalized* characteristic function, then:

- With the Shapley value, a_1 contributes *twice* the difference between $v(\{a_2, a_3\})$ and $v(\{a_1, a_2, a_3\})$.
- With the NR value, a_1 contributes *once* the difference between $v_g(a_2, a_3)$ and $v_g(a_2, a_3, a_1)$, and contributes *once* the difference between $v_g(a_3, a_2)$ and $v_g(a_3, a_2, a_1)$.
- With the SB value, a_1 contributes *twice* the difference between the *average value* of a coalition consisting of a_2, a_3 and of a coalition consisting of a_1, a_2, a_3, i.e., difference between $\frac{v_g(a_2,a_3)+v_g(a_3,a_2)}{2}$ and $\frac{v_g(a_1,a_2,a_3)+v_g(a_1,a_3,a_2)+v_g(a_2,a_1,a_3)+v_g(a_2,a_3,a_1)+v_g(a_3,a_1,a_2)+v_g(a_3,a_2,a_1)}{6}$.

One of the interesting applications of those two extensions is the recent body of literature that uses game theoretic solution concepts to compute centralities of nodes in networks [8, 9, 10]. In summary, by defining a coalitional game with players being nodes of a network, and then computing a solution for such a game, it is possible to obtain a measure of importance for individual nodes. In this context, both the NR and SB values were used by del Pozo et al. [8] to study the centrality of nodes in *directed social networks*. These networks have recently raised increasing attention as they can be used to model a variety of situations, ranging from terrorist groups [11] to the spread of contagious diseases [12]. The crucial characteristics of these real-life network applications is that a relationship between two nodes connected by an edge is asymmetric, i.e., the edge is directed. Consequently, in many cases, the worth of a coalition in a game defined over such a network should depend not only on its members but also on the order in which they were incorporated to this coalition. Del Pozo et al. took this into account by defining a *generalized* characteristic function game over a network (instead of a characteristic function game) and considering both NR and SB values as centrality measures.

Although there have been a number of game-theoretic works (including the NR and SB values) on generalized characteristic function games, the *implementational* aspects of these games have not been yet studied. This research challenge can be summarized as follows. Given a desired solution to a coalitional game (such as the Shapley value), the issue of implementation deals with creating a set of rules (a mechanism) that incentivizes self-interested players to reach the desired solution as a result of equilibrium behavior. Although there exist various mechanisms implementing the Shapley value and some of its various extensions, no mechanism for coalitional games with ordered coalitions has been proposed to date.

Against this background, in this article we present an implementation of the NR value and the SB value. We build upon the Simple Demand Commitment Games by Dasgupta and Chiu [13] in which the Shapley value is implemented in expectations. We call our two refinements for the generalized characteristic function games the *Ordered Demand Commitment Games*. They implement both the NR value and the SB value in expectations.

The remainder of the article is organized as follows. In Section 2, we discuss the related literature. Section 3 provides notation and formal definitions of the Shapley value and its two extensions to generalized characteristic function games (i.e., the NR and SB values). In Section 4 we describe our mechanisms for implementing both values. Conclusions and possible future work follow.

2 Related Work

The issue of implementing the Shapley value has been studied in the literature by a number of authors. Gul [14] introduced a model of a transferable-utility economy where two players make bilateral offers at random meetings. Assuming that the game is strictly convex, with the time interval between meetings becoming arbitrarily small, the Shapley value emerges as the limit for the expected payoff of each player in a Stationary Subgame-Perfect Nash Equilibrium (SSPNE). This result holds also for strictly super-additive games but only for the SSPNE reached by an immediate agreement ([14, 15]). A simplified mechanism of this kind was introduced by Evans [16].

Two alternative mechanisms were later on introduced by Dasgupta and Chiu [13] and by Pérez-Castrillo and Wettstein [17]. Dasgupta and Chiu [13] proposed a mechanism called the Simple Demand Commitment Game (SDCG). Assuming that the characteristic function is strictly convex, the mechanism starts by randomly choosing an order in which players are allowed to move. Then, the first player in the chosen order makes a move which may or may not end the game. If the game does not end, then the second player in the order makes another move (which again may or may not end the game), and so on. The move that each player a_i makes is to select one of the following two options: (1) demand a payoff d_i that a_i will accept in return for joining any coalition, or (2) create a coalition consisting of a_i and a (possibly empty) subset of his choice out of the players that precede him in the order, which ends the game and forces every non-member of that coalition to form a singleton coalition.

The above mechanism by Dasgupta and Chiu implements the Shapley value in expected terms. To avoid this limitation, Pérez-Castrillo and Wettstein [17] proposed an alternative mechanism by which the Shapley value emerges in all equilibria. Furthermore, compared to Dasgupta and Chiu's mechanism, which requires strict convexity, Pérez-Castrillo and Wettstein's mechanism requires the characteristic function is zero-monotonic, which is a weaker condition. In more detail, the mechanim by Pérez-Castrillo and Wettstein involves three steps. In Step 1 players bid by offering each other transfers and the stake is to become a proposer, that is to have the sole right to divide the payoff from the game. In Step 2, the winner (i.e. the highest net bidder) pays the transfers promised to other players, and then proposes the division of the game's payoff among the players in the game. In Step 3, these players either accept or reject the proposal. If the offer is rejected, the proposer is obliged to leave the mechanism and form a singleton coalition. The remaining players follow the same procedure but for the now-smaller game. In essence, this mechanism hinges upon the *balanced contribution property* of the Shapley value, which basically states that any player is worth the same to any other player in the game. More formally:

$$\phi_i(N,v) - \phi_i(N_{-j},v) = \phi_j(N,v) - \phi_j(N_{-i},v), \tag{1}$$

where $\phi_i(N,v)$ denotes the Shapley value of player a_i in the coalitional game with player set N and value function v, while, $\phi_i(N_{-i},v)$ denotes the Shapley value of player a_i in the coalitional game with player set $N\backslash\{a_i\}$ and the same value function v. Pérez-Castrillo and Wettstein showed how this property allows for the construction of a mechanism in which the Shapley value emerges as a result of equilibrium behavior. Unfortunately, it can be very easily demonstrated the balanced contribution property is not met by the Nowak and Radzik value.

A number of follow-up works have built upon Pérez-Castrillo and Wettstein's mechanism. In particular, the version of the mechanism that implements the Ordinal Shapley value for $n = 3$ was proposed by Pérez-Castrillo and Wettstein [18]. To implement the Owen value, Vidal-Puga and Bergantiños [19] added a fourth step—a bidding phase to become leaders of *a priori* given coalitions. A different fourth step was proposed by Ju and Wettstein [20] in order to implement the extension of the Shapley value to games with externalities by Pham do and Norde [21]. A similar approach followed in Macho-Stadler et al. [22, 23]. Van den Brink and Funaki [24] introduced a discounting parameter to implement the discounted Shapley value.

Finally, we mention other extensions of the Shapley value for generalized characteristic function games developed after the Nowak-Radzik and Sánchez-Bergantiños values. In particular, a family of weighted Shapley values was studied by Bergantiños and Sánchez [25]. Furthermore, a parametric family of values (including both the Nowak-Radzik and Sánchez-Bergantiños values) was analyzed in a network context by del Pozo et al. [8].

3 Preliminaries

We begin by describing the basic notation (Appendix B provides a comprehensive summary). Let $N = \{a_1, \ldots, a_n\}$ be the set of players in a coalitional game. Denote by 2^N the set of all subsets of N. An element of 2^N is a *coalition*. An arbitrary coalition will often be denoted C or D. The coalition involving all players in the game will be called the *grand coalition*. A *characteristic function* v is a mapping $v : 2^N \to \mathbb{R}$, i.e., it assigns to every coalition $C \subseteq N$ a real number representing its value. We will assume that $v(\varnothing) = 0$. A *game in characteristic function form* is a pair (N, v). When there is no risk of confusion, we will sometimes simply write v instead of (N, v).

For each coalition $C \in 2^N \backslash \{\varnothing\}$, denote by $\Pi(C)$ the set of all possible permutations of the players in C. Any such permutation will be called an *ordered coalition*. An arbitrary ordered coalition will often be denoted as T or S, while the set of all such coalitions will be denoted $\mathcal{T}$. That is, $\mathcal{T} = \bigcup_{C \in 2^N} \Pi(C)$. A *generalized characteristic function* v_g is a mapping $v_g : \mathcal{T} \to \mathbb{R}$, where it is assumed that $v_g(\varnothing) = 0$. A *game in generalized characteristic function form* is a tuple (N, v_g), and will sometimes be denoted by v_g alone. For some coalition $D \subseteq N$ we will denote by $\mathcal{T}_{-D}$ the set of all ordered coalitions not containing players from D, formally: $\mathcal{T}_{-D} = \bigcup_{C \in 2^{N \backslash D}} \Pi(C)$. Sometimes, a_i will be used implicitely as (a_i).

We will sometimes refer to the members of an ordered coalition T using their names, e.g., write $T = (a_5, a_2, a_3)$, while other times we may refer to them using a lower case of the same letter: $T = (t_1, \ldots, t_{|T|})$, meaning that t_i is the i^{th} agent in T. Furthermore, given two disjoint ordered coalitions, $T = (t_1, \ldots, t_{|T|}) \in \mathcal{T}$ and $S = (s_1, \ldots, s_{|S|}) \in \mathcal{T}$, we write $(T, S)^k$ to denote the ordered coalition that results from inserting S at the k^{th} position in T. That is, $(T, S)^k = (t_1, \ldots, t_{k-1}, s_1, \ldots, s_{|S|}, t_k, \ldots, t_{|T|})$. With a slight abuse of notation, we write $(T, a_i)^k$ to denote $(T, (a_i))^k$. Furthermore, we write (a_i, T), and (T, a_i), to denote the ordered coalition that results from inserting a_i to T as the first player, and the last player, respectively.

For every coalition $C \subseteq N$ and every permutation $\pi = \{\pi_1, \pi_2, \ldots, \pi_{|\pi|}\} \in \Pi(C)$, we introduce a function $inv(\pi)$ that returns the inverse of π. Formally, $inv : \bigcup_{C \subseteq N} \Pi(C) \to \bigcup_{C \subseteq N} \Pi(C)$ is given by: $inv(\pi) = (\pi_{|\pi|}, \ldots, \pi_2, \pi_1)$. For instance, for $\pi = (a_3, a_1, a_5, a_6)$, we have $inv(\pi) = (a_6, a_5, a_1, a_3)$. Furthermore, given a permutation $\pi \in \Pi(N)$ and a coalition $C \subseteq N$, with a slight abuse of notation we will denote by $\pi(C)$ the ordered coalition consisting of all the players in C ordered according to π, i.e., it is the ordered coalition that results after removing from π every player in $N \backslash C$. For example, given $\pi = (a_2, a_1, a_4, a_3)$, and $C = \{a_1, a_2, a_3\}$, we have $\pi(C) = (a_2, a_1, a_3)$. Moreover, given a generalized game (N, v_g), and a permutation $\pi \in \Pi(N)$, we denote by $(N, v_{g,\pi})$ the characteristic function game in which, $\forall C \subseteq N$:

$$v_{g,\pi}(C) = v_g(\pi(C)). \tag{2}$$

For any (N, v_g), we also introduce the characteristic function game $(N, \bar{v}_g)$ which we call the *average game* of (N, v_g). In this game, for $\forall C \subseteq N$:

$$\bar{v}_g(C) = \frac{1}{|\Pi(C)|} \sum_{T \in \Pi(C)} v_g(\pi(C)). \tag{3}$$

We will call $\bar{v}_g$, the *average characteristic function* of (N, v_g).

Next, we extend the notion of a subset to ordered sets.

Definition 1. *For any two ordered coalitions $S = (s_1, \ldots, s_{|S|}) \in \mathcal{T}$ and $T = (t_1, \ldots, t_{|T|}) \in \mathcal{T}$, we say that T is a **subset** of S, and write $T \vec{\subseteq} S$, if and only if T is a subsequence of S, i.e., the following two conditions hold:*

- *Every members of T is a member of S. More formally:*

$$\forall t_i \in T, \ \exists s_k \in S : s_k = t_i.$$

- *For any two players, $t_i, t_j \in T$, if t_i appears before t_j in T, then t_i also appears before t_j in S. More formally:*

$$\forall t_i, t_j \in T : i < j, \exists s_k, s_w \in S : k < w \text{ and } s_k = t_i \text{ and } s_w = t_j.$$

Following convention, we say that T is a strict *subset of S, and write $T \vec{\subset} S$ (instead of $T \vec{\subseteq} S$), if the above two conditions are met, and $T \neq S$.*

Now, we are ready to introduce the following definitions:

Definition 2. *A characteristic function game (N, v) is said to be (strictly) **zero monotonic** if, for all $a_i \in N$ and $C \subseteq N \backslash \{a_i\}$, the following holds:*

$$v(C) + v(\{a_i\})(<) \leqslant v(C \cup \{a_i\}).$$

Definition 3. *A generalized characteristic function game (N, v_g) is said to be (strictly) **zero monotonic** if, for all $C \subseteq N$, for all $a_i \in C$, and for all $T \in \Pi(C_{-i})$*

$$v_g(T) + v_g((a_i))(<) \leqslant v_g((T, a_i)).$$

A stricter condition than zero-monotonicity is convexity:

Definition 4. *A characteristic function game (N, v) is said to be (strictly)* ***convex*** *if, for every two coalitions $C, D : D \subset C$ and for every $a_i \in N \backslash C$, the following holds:*

$$v(C \cup \{a_i\}) - v(C)(>) \geqslant v(D \cup \{a_i\}) - v(D).$$

We extend the notion of convexity to the generalized characteristic function games as follows:

Definition 5. *A generalized characteristic function game (N, v_g) is said to be (strictly)* ***convex*** *if, for every two ordered coalitions $S, T : T \overrightarrow{\subset} S$ and for every $a_i \in N \backslash S$, we have:*

$$v_g((S, a_i)^s) - v_g(S)(>) \geqslant v_g((T, a_i)^t) - v_g(T),$$

whenever $(T, a_i)^t$ is a subset of $(S, a_i)^s$. More formally, the inequality holds for every $t \in \{1, \ldots, |T| + 1\},\ s \in \{1, \ldots, |S| + 1\} :\ (T, a_i)^t \overrightarrow{\subset} (S, a_i)^s$.

Now, we briefly describe the Shapley value for characteristic function games, and then present its extensions to *generalized* characteristic function games. The Shapley value was proposed as a normative scheme for dividing the value of the game *fairly* among the players. In more detail, the Shapley value of a player $a_i \in N$, denoted $\phi_i(N, v)$, is a_i's *share* of the grand coalition's payoff, which is computed as the average *marginal contribution* of that player over all possible joining orders (assuming that the agents have joined the game sequentially, one agent at a time). Formally:

$$\phi_i(N, v) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \Delta_v(C_{\overleftarrow{\pi_i}}, a_i), \tag{4}$$

where $\Delta_v(C_{\overleftarrow{\pi_i}}, a_i)$ is the marginal contribution of a player a_i to a coalition $C_{\overleftarrow{\pi_i}}$ consisting of all the players that are in permutation π before a_i. Formally:

$$\Delta_v(C_{\overleftarrow{\pi_i}}, a_i) = v(C_{\overleftarrow{\pi_i}} \cup \{a_i\}) - v(C_{\overleftarrow{\pi_i}}). \tag{5}$$

Importantly, as visible from Equation (4), if $\pi \in \Pi(N)$ was selected uniformly at random, the Shapley value of player a_i would be the expected marginal contribution of a_i to $C_{\overleftarrow{\pi_i}}$. That is, $\phi_i(N, v) = \mathbb{E}[\Delta_{C_{\overleftarrow{\pi_i}}, i}]$, where $\mathbb{E}[\cdot]$ is the expectation operator.

It is possible to rewrite Equation (4) as follows:

$$\phi_i(N, v) = \sum_{\pi \in N \backslash \{a_i\}} \frac{(|N| - |C| - 1)!(|C|!)}{|N|!} v(C \cup \{a_i\}) - v(C). \tag{6}$$

This is more computationally efficient than Equation (4), because the sum is over coalitions, not permutations. When there is no risk of confusion, instead of $\phi_i(N, v)$, we will write $\phi_i(v)$ or ϕ_i for brevity. This also concerns the extensions of the Shapley value that will be presented later on in this section.

The Shapley value is "fair" in the sense that it is the unique solution that has the following axioms:

Symmetry: The payoffs do not depend on the players' names. That is, $\phi(\pi(v)) = \pi(\phi)(v)$ for every game v and permutation $\pi \in \Pi(N)$.

Null Player: The players that make no contribution should receive nothing. In other words, we have $\big(\forall C \subseteq N, \Delta_v(C_{\overleftarrow{\pi_i}}, a_i) = 0\big) \Rightarrow \big(\phi_i(N) = 0\big)$.

Efficiency: The entire payoff of the grand coalition should be distributed among its members. That is, $\sum_{a_i \in N} \phi_i(N) = v(N)$.

Additivity: Given three games, (N, v_1), (N, v_2) and (N, v_3), where $v_1(C) = v_2(C) + v_3(C)$, it holds that, for all $C \subseteq N$, the payoff of a player in (N, v_1) is the sum of its payoffs in (N, v_2) and in (N, v_3).

Whereas these four axioms uniquely determine the Shapley value for characteristic function games, the situation is more complex for generalized games, because a player's marginal contribution (and consequently the *symmetry* and *null-player* axioms) depends on where the new player in the coalition is placed. In this respect, Nowak and Radzik [2] developed an extension of the Shapley value by making perhaps the most natural assumption that the marginal contribution of a player is computed when this player is placed *last* in the coalition. Let us denote this marginal contribution of a_i to $T \in \mathcal{T}(N\backslash\{a_i\})$ in game v_g (according to Nowak and Radzik's definition) as $\Delta_{v_g}^{NR}(T, a_i)$. Then:

$$\Delta_{v_g}^{NR}(T, a_i) = v_g((T, a_i)) - v_g(T). \tag{7}$$

In what follows, for any ordered coalition, T, let $T(a_i)$ denote the sequence of players in T that appear before a_i (if $a_i \notin T$ then $T(a_i) = T$). For example, given $T = (a_1, a_3, a_4, a_6)$, we have $T(a_4) = (a_1, a_3)$. Using this notation, the *Nowak-Radzik value* (or the *NR value* for short) is defined as follows:

$$\phi_i^{NR}(N, v_g) = \frac{1}{|N|!} \sum_{T \in \Pi(N)} \Delta_{v_g}^{NR}(T(a_i), a_i) = \mathbb{E}[\Delta_{v_g}^{NR}(T(a_i), a_i)]. \tag{8}$$

This can be written differently as follows:

$$\phi_i^{NR}(N, v_g) = \sum_{C \subseteq N_{-i}} \sum_{T \in \Pi(C)} \frac{(|N| - |T| - 1)!}{|N|!} [v_g((T, a_i)) - v_g(T)]. \tag{9}$$

The NR value is the unique value that satisfies the following "fairness" axioms:

Efficiency: $\sum_{a_i \in N} \phi_i^{NR}(v_g) = \frac{1}{|N|!} \sum_{T \in \Pi(N)} v_g(T)$.

Null-Player: $\forall a_i \in N$, if $v_g(T) = v_g((T, a_i))\ \forall T \in \mathcal{T} : a_i \notin T$, then $\phi_i^{NR}(v_g) = 0$.

Additivity: $\phi^{NR}(v_g + v_g') = \phi^{NR}(v_g) + \phi^{NR}(v_g')$ for any two functions, v_g and v_g'.

Sánchez and Bergantiños [7] developed an alternative extension of the Shapley value based on the definition of the marginal contribution, where, instead of assuming that this player will be placed last, the authors take the average over all possible positions in which the player can be placed:

$$\Delta_{v_g}^{SB}(T, a_i) := \frac{1}{(|T|+1)} \sum_{l=1}^{|T|+1} [v_g((T, a_i)^l) - v_g(T)]. \tag{10}$$

The *Sánchez-Bergantiños value* (or *SB value* for short) is then computed as:

$$\phi_i^{SB}(N, v_g) = \frac{1}{|N|!} \sum_{T \in \Pi(N)} \Delta_{v_g}^{SB}(T, a_i) = \mathbb{E}[\Delta_{T(a_i)}^{SB}]. \tag{11}$$

This also can be rewritten differently as follows:

$$\phi_i^{SB}(N, v_g) = \sum_{C \subseteq N_{-i}} \sum_{T \in \Pi(C)} \frac{(|N|-|T|-1)!}{|N|!(|T|+1)} \sum_{l=1}^{|T|+1} [v_g((T, a_i)^l) - v_g(T)]. \tag{12}$$

As noted by Sánchez and Bergantiños [7], their value for v_g is equivalent to the Shapley value of the average game of v_g (see Equation 3 for the definition of the average game), i.e.,

$$\phi_i^{SB}(N, v_g) = \phi_i(N, \bar{v}_g) = \mathbb{E}[\Delta_{\bar{v}_g}(C_{\overleftarrow{\pi_i}}, a_i)]. \tag{13}$$

The SB value is the unique value that satisfies NR's efficiency and additivity axioms and the following axioms:

Null-Player If $\forall T \in \mathcal{T}\ \forall l \in \{1, .., |T|+1\}\ :\ v_g((T, a_i)^l) = v_g(T)$, then $\phi_i^{SB}(v_g) = 0$.

Symmetry If $\forall T \in \mathcal{T}_{-\{i,j\}} \forall l \in \{1, .., |T|+1\}\ :\ v_g((T, a_i)^l) = v_g((T, a_j)^l)$, then $\phi_i^{SB}(v_g) = \phi_j^{SB}(v_g)$.

The difference between the NR and SB values is illustrated in the following example:

Example 1. *Consider a game with an ordered coalition $T^* \in \Pi(N)$ such that $v_g(T) = 1$ if $T = T^*$ and $v_g(T) = 0$ otherwise. Then, the average value of the grand coalition, taken over all possible orders, which is $\frac{1}{n!}$, needs to be distributed among the players. Using the NR value, we get $\phi_t^{NR}(N) = \frac{1}{n!}$, where a_t is the last player in the ordered coalition T^*, and we get $\phi_i^{NR}(N) = 0$ for all $a_i \in N \backslash \{a_t\}$. In contrast, using the SB value, we get $\phi_i^{SB}(N) = \frac{1}{n! \cdot |T'|} = \frac{1}{n! \cdot n}$ for all $a_i \in N$. As can be seen, in this example, the NR value rewards the last player in the order, whereas the SB value rewards all players equally.*

Having introduced the Shapley value and its extensions to generalized characteristic function games, in the following section we consider the issue of implementation.

4 Implementation

Among the many deeply-studied aspects of the Shapley value is whether there exists a set of rules (or a mechanism) that incentivizes self-interested players to adopt the Shapley value as a result of equilibrium behavior.[1] In this section we propose a mechanism to implement the NR and SB values in Subgame-Perfect Nash Equilibria (SPNE).[2] We build upon the mechanism by Dasgupta and Chiu [13].

Given a characteristic function game v, Dasgupta and Chiu's mechanism is called the Simple Demand Commitment Game, dentoed by $SDCG(v)$. The mechanism proposed in this section modifies it to handle ordered coalitions, i.e., to handle a generalized characteristic function game v_g. As such, we call this mechanism *Ordered Demand Commitment Game*. It has two versions, one for the NR value (called $ODCG^{NR(v_g)}$) and the other for the SB value (called $ODCG^{SB}(v_g)$). Section 4.1 presents $ODCG^{NR}(v_g)$, while Section 4.2 presents $ODCG^{SB}(v_g)$. Section 4.2.1 proves that each mechanism implements its respective value, and that each of the aforementioned strategies is, in fact, an Subgame-Perfect Nash Equilibria (SPNE).

4.1 The $ODCG^{NR}(v_g)$ mechanism

The mechanism $ODCG^{NR}(v_g)$ has two main steps:

- **Step 1:** An order of players is chosen uniformly at random out of all possible orders. Without loss of generality, let the chosen order be $\pi = (a_1, \dots, a_n)$.
- **Step 2:** The first player in π (i.e., a_1) makes the first move, then the second player in π (i.e., a_2) makes the second move (unless a_1 has terminated the game), then the third player in π (i.e., a_3) makes the third move (unless the game has been terminated before his turn), and so on. The move of every $a_i : 1 \leqslant i < n$ involves choosing one of the following two options:
  - **Option 1:** Specify a "demand" $d_i \in \mathbb{R}$—an amount of utility that a_i will accept in return for joining any coalition. The mechanism then proceeds to the subsequent player in the order, i.e., a_{i+1}.
  - **Option 2:** Select a subset $C \subseteq \{a_1, \dots, a_{i-1}\}$ that a_i wants to join. This terminates the game with the following outcome: Coalition $inv(\pi(C \cup \{a_i\}))$ forms, and its payoff is divided as follows: Every $a_k \in C$ receives d_k, while a_i receives:

$$v_g(inv(\pi(C \cup \{a_i\}))) - \sum_{a_k \in C} d_k. \tag{14}$$

[1] This is part of the *Nash program*, which tries to provide a non-cooperative foundation for cooperative solution concepts [26].

[2] The SPNE of a game G are all strategy proles s such that for any subgame G' of G, the restriction of s to G' is a Nash Equilibrium of G'. For more details see Shoham and Leyton-Brown [27].

In other words, a_i pays the members their demands, and takes the surplus for himself. As for non-members, every $a_j \in N \backslash (C \cup \{a_i\})$ is left with no choice but to form the singleton coalition $\{a_j\}$ and receive the payoff $v_g((a_j))$.

Player a_n on the other hand has only one choice, which is **Option 2**.

Note that the above mechanism is a game of perfect information, as the chosen order is made publicly known before any player makes a move. **Step 2** of $ODCG^{NR}(v_g)$ and $ODCG^{SB}(v_g)$ will be denoted by $ODCG^{NR}_{\pi}(v_g)$ and $ODCG^{SB}_{\pi}(v_g)$, where π is the order chosen in **Step 1**.

4.2 The $ODCG^{SB}(v_g)$ mechanism and the $\sigma^{SB}_{\pi,i}$ strategy

The $ODCG^{SB}(v_g)$ mechanism is identical to $ODCG^{NR}(v_g)$ except for the following difference. In **Option 2**, the payoff of player a_i in Equation (14) becomes:

$$v_g(\tilde{\pi}(C \cup \{a_i\})) - \sum_{a_k \in C} d_k,$$

where $\tilde{\pi}(C \cup \{a_i\})$ is an ordered coalition chosen uniformly at random from the set $\Pi^{C \cup \{a_i\}}$. This means a_i can choose the identities (but not the order) of the agents who will join him in the same coalition. The order will be chosen randomly by the mechanism, only after the members are chosen by a_i.

Table 1 summarizes the differences between the $SDCG$ mechanism proposed by Dagupta and Chiu [13] and the $ODCG^{NR}$ and $ODCG^{SB}$ mechanisms proposed in this article, where p_k denotes the payoff of a_k.

4.2.1 Properties of the Mechanisms

Our key results with regards to the $ODCG^{NR}(v_g)$ and $ODCG^{SB}(v_g)$ mechanisms are presented in the following theorem.

Theorem 1. *Every SPNE of $ODCG^{NR}(v_g)$ and $ODCG^{SB}(v_g)$ has payoffs equal to the NR value, and the SB value, respectively.*

Proof. We start by recalling Equations (4), (8), and (13) in which either the Shapley value, the NR value or the SB value, respectively, are presented as the expected marginal contribution of player a_i in permutation π, where $\pi \in \Pi(N)$ is selected uniformly at random. While this general functional form is the same for all three values, their differences stem:

Table 1: A comparison between the $SDCG$ mechanism by Dasgupta and Chiu [13] and our $ODCG^{NR}(v_g)$ and $ODCG^{SB}(v_g)$ mechanisms. In **Step 1** of all three mechanisms, a permutation of players π is chosen randomly to determine the order of moves (the i^{th} player in π makes the i^{th} move). Without loss of generality, the table assumes $\pi = (a_1, a_2, \ldots, a_n)$. In **Step 2**, the move of a_i is to make a choice between (1) demanding d_i in return for joining any coalition requested by a subsequent player, and (2) forming a coalition with (some of) the previous players and ending the game. The main difference is in the way the mechanisms impose an order on the resulting multi-player coalition.

	Option 1 of a_i	**Option 2** of a_i	Coalition created by the mechanism	Payoffs
$SDCG_\pi(v)$	demand d_i	choose a subset $C \in \{a_1, \ldots, a_{i-1}\}$	$C \cup \{a_i\}$	$\forall a_k \in C : p_k = d_k$ $p_i = v(C \cup \{a_i\}) - \sum_{a_k \in C} d_k$ $\forall a_k \in N \backslash \{C \cup \{a_i\}\} : p_k = v(\{a_k\})$.
$ODCG^{NR}_\pi(v_g)$	demand d_i	choose a subset $C \in \{a_1, \ldots, a_{i-1}\}$	$\underbrace{inv(\pi(C \cup \{a_i\}))}_{C \cup \{a_i\} \text{ ordered as inverse of } \pi}$	$\forall a_k \in C : p_k = d_k$ $p_i = v_g(inv(\pi(C \cup \{a_i\}))) - \sum_{a_k \in C} d_k$ $\forall a_k \in N \backslash \{C \cup \{a_i\}\} : p_k = v_g((a_k))$.
$ODCG^{SB}_\pi(v_g)$	demand d_i	choose a subset $C \in \{a_1, \ldots, a_{i-1}\}$	$\underbrace{\tilde{\pi}(C \cup \{a_i\})}_{(C \cup \{a_i\} \text{ ordered at random})}$	$\forall a_k \in C : p_k = d_k$ $p_i = v_g(\tilde{\pi}(C \cup \{a_i\})) - \sum_{a_k \in C} d_k$ $\forall a_k \in N \backslash \{C \cup \{a_i\}\} : p_k = v_g((a_k))$.

- from the different underlying value function—in the case of the Shapley value this is the characteristic function v, in the case of the NR value this is the generalized characteristic function v_g, and in the case of the SB value this is the average characteristic function for v_g, i.e., it is $\bar{v}_g$; and/or
- from the different definition of the marginal contribution—in the case of the Shapley value and the SB value we have $\Delta_v(C_{\overleftarrow{\pi_i}}, a_i)$ and $\Delta_{\bar{v}_g}(C_{\overleftarrow{\pi_i}}, a_i)$, respectively (Equation (5) for the value function v and $\bar{v}_g$), while for the NR value we have $\Delta^{NR}_{v_g}(T, a_i)$ (Equation (7)).

Step 1 of both our mechanisms is the same as **Step 1** of the mechanism by Dasgupta and Chiu [13]: a permutation π is chosen uniformly at random. Therefore, what we need to show for each value is that our refinements account for the differences in the value function and the marginal contribution as outlined above. To this end, let us consider the following three lemmas:

LEMMA 4.1.1. *Given (N, v_g), let $\pi \in \Pi(N)$ be an order of players chosen uniformly at random in **Step 1** of $ODCG^{NR}(v_g)$ ($ODCG^{SB}(v_g)$). Then, for every player $a_i \in N$, **Step 2** of the mechanism, i.e., $ODCG^{NR}_\pi(v_g)$ ($ODCG^{NR}_\pi(v_g)$), is strategically equivalent to $SDCG_\pi(v_{g,\pi})$ ($SDCG_\pi(\bar{v}_g)$).*

Proof. We consider $ODCG^{NR}(v_g)$ first. By rules of this mechanism (see Table 1), any coalition $C \cup \{a_i\}$ chosen by a_i in **Option 2** will be created as an ordered coalition $inv(\pi(C \cup \{a_i\}))$.

Thus, the choices offered to a_i by $ODCG^{NR}(v_g)$, i.e., every $inv(\pi(C \cup \{a_i\}))$ in the generalized characteristic function game v_g, are in fact equivalent to the choices offered to a_i by $SDCG(v_{g,inv(\pi)})$, i.e., every $C \cup \{a_i\}$ in the characteristic function game $v_{g,inv(\pi)}$. This shows that **Step 2** of $ODCG^{NR}(v_g)$ is strategically equivalent to **Step 2** of $SDCG(v_{g,inv(\pi)})$.

Turning now to $ODCG^{SB}(v_g)$, by rules of this mechanism (see again Table 1), any coalition $C \cup \{a_i\}$ chosen by a_i in **Option 2** will be created as an ordered coalition $(a_i, \tilde{\pi}(C))$. Recall that $\tilde{\pi}(C)$ denotes a randomly ordered coalition made of players in C. Since player a_i has to select C without knowing how it will be ordered, his rational behaviour is to consider the expected value of $C \cup \{a_i\}$ over all possible orders of C, bearing in mind that he will be placed in the first position of the ordered coalition (as per the rules of the mechanism). This expected value is:

$$\frac{1}{|C|!} \sum_{\tilde{\pi} \in \Pi(C)} v_g(\tilde{\pi}(C \cup \{a_i\})),$$

which is precisely $\bar{v}_g(C \cup \{a_i\})$ (see Section 3). Thus, the choices offered to a_i by $ODCG^{SB}(v_g)$, i.e., every $(a_i, \tilde{\pi}(C))$ in the generalized characteristic function game v_g, are equivalent to the choices offered to a_i by $SDCG(\bar{v}_g)$, i.e., every $C \cup \{a_i\}$ in the characteristic function game $\bar{v}_g$. This shows that **Step 2** of $ODCG^{SB}(v_g)$ is strategically equivalent to **Step 2** of $SDCG(\bar{v}_g)$. □

LEMMA 4.1.2. *Given a (strictly) convex ordered game (N, v_g), and a permutation $\pi \in \Pi(N)$, the game $(N, v_{g,\pi})$ is (strictly) convex.*

Proof. We need to show that:

$$v_{g,\pi}(C \cup \{a_i\}) - v_{g,\pi}(C) \ (>) \geqslant \ v_{g,\pi}(D \cup \{a_i\}) - v_{g,\pi}(D), \quad \text{where } a_i \in N \backslash C \text{ and } D \subset C \subset N. \tag{15}$$

To this end, observe that every member of D appears in C, and if a player, a_i, appears before another, a_j, in $\pi(D)$, then it will also appear before it in $\pi(C)$, as both coalitions are ordered according to π. Therefore, based on Definition 1, we have: $\pi(D) \vec{\subset} \pi(C)$. By a similar reasoning, we have $\pi(C \cup \{a_i\}) \vec{\supset} \pi(D \cup \{a_i\})$. This, as well as the fact that v_g is convex, implies the following (based on Definition 5):

$$v_g(\pi(C \cup \{a_i\})) - v_g(\pi(C)) \ (>) \geqslant \ v_g(\pi(D \cup \{a_i\})) - v_g(\pi(D)). \tag{16}$$

Moreover, by definition, we have $v_{g,\pi}(C) = v_g(\pi(C))$ for every $C \subseteq N$. This, together with Equation (16), imply that Equation (15) holds. □

LEMMA 4.1.3. *Given a (strictly) convex ordered game (N, v_g), and a permutation $\pi \in \Pi(N)$, the game $(N, \bar{v}_g)$ is (strictly) convex.*

Proof. We know from Lemma 4.2.1 that, for all $\pi \in \Pi(N)$ and all $a_i \in N$, we have:

$$v_{g,\pi}(C \cup \{a_i\}) - v_{g,\pi}(C) \ (>) \geqslant \ v_{g,\pi}(D \cup \{a_i\}) - v_{g,\pi}(D),$$

for every $D \subset C \subset N \backslash \{a_i\}$. Thus:

$$\frac{1}{|N|!}\sum_{\pi\in\Pi(N)}\Big(v_{g,\pi}(C\cup\{a_i\})-v_{g,\pi}(C)\Big)\ (>)\geqslant\ \frac{1}{|N|!}\sum_{\pi\in\Pi(N)}\Big(v_{g,\pi}(D\cup\{a_i\})-v_{g,\pi}(D)\Big). \tag{17}$$

Now observe that, for every coalition $C\subseteq N$, we have:

$$\frac{1}{|N|!}\sum_{\pi\in\Pi(N)}\big(v_\pi(C)\big)=\frac{1}{|C|!}\sum_{\pi\in\Pi(C)}\big(v_\pi(C)\big)=\bar{v}_g(C). \tag{18}$$

Equations (17) and (18) mean that:

$$\bar{v}_g(C\cup\{a_i\})-\bar{v}_g(C)\ (>)\geqslant\ \bar{v}_g(D\cup\{a_i\})-\bar{v}_g(D).$$

□

Next, building upon the above three lemmas, as well as the result of Dasgupta and Chiu [13], we prove the correctness of Theorem 1.

Dasgupta and Chiu [13] showed that, for a convex characteristic function game v, all the SPNE of their mechanism, $SDCG(v)$, result in the Shapley value in expectation. They also showed that in the equilibrium any player a_i in the randomly chosen order π (except for the last player) makes a demand that equals his contribution to the coalition consisting of all subsequent players in π (we will denote this coalition by $C_{\overrightarrow{\pi_i}}$). For instance, given $(\{a_1,a_2,a_3\},v)$ and $\pi=(a_2,a_1,a_3)$, player a_2 demands $d_2=\Delta_v(C_{\overrightarrow{\pi_2}},a_2)=v(\{a_1,a_3\}\cup\{a_2\})-v(\{a_1,a_3\})$, and player a_1 demands $d_1=\Delta_v(C_{\overrightarrow{\pi_1}},a_1)=v(\{a_3\}\cup\{a_1\})-v(\{a_3\})$. The last player in π, i.e., a_3, forms the grand coalition, $\{a_1,a_2,a_3\}$, and satisfies the demands of a_1 and a_2, leaving him with a payoff equal to his marginal contribution to the empty set. That is, a_3 receives:

$$v(\{a_1,a_2,a_3\})-d_2-d_1=v(\{a_3\}),$$

or, equivalently, $\Delta_v(C_{\overrightarrow{\pi_3}},a_3)=v(\varnothing\cup\{a_3\})-v(\varnothing)$.

Then, since the following holds:

$$\mathbb{E}[\Delta_{C_{\overrightarrow{\pi_i}},i}]=\frac{1}{|N|!}\sum_{\pi\in\Pi(N)}\Delta_v(C_{\overrightarrow{\pi_i}},a_i)=\frac{1}{|N|!}\sum_{\pi\in\Pi(N)}\Delta_v(C_{\overleftarrow{\pi_i}},a_i)m=\mathbb{E}[\Delta_{C_{\overleftarrow{\pi_i}},i}], \tag{19}$$

the $SDCG(v)$ mechanism implements the Shapley value in expectation (see Equation (4)).

From the above result of Dasgupta and Chiu, as well as Lemmas 4.2.1 and 4.2.1, it follows that, for a convex generalized characteristic function, v_g, all SPNE of our mechanism $ODCG^{NR}(v_g)$ result in expectation in the following value:

$$\phi_i^*(N,v_g)=\frac{1}{|N|!}\sum_{\pi\in\Pi(N)}\Delta_{v_{g,inv(\pi)}}(C_{\overrightarrow{\pi_i}},a_i), \tag{20}$$

while all SPNE of our mechanism $ODCG^{SB}(v_g)$ result in expectation in:

$$\phi_i^{**}(N, v_g) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \Delta_{\bar{v}_g}(C_{\overrightarrow{\pi_i}}, a_i). \tag{21}$$

Equations (20) and (21) imply that, in order to prove Theorem 1, it suffices to prove that, the following two equations hold for all $a_i \in N$:

$$\phi_i^{*}(N, v_g) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \Delta_{v_{g,inv(\pi)}}(C_{\overrightarrow{\pi_i}}, a_i) = \frac{1}{|N|!} \sum_{T \in \Pi(N)} \Delta_{v_g}^{NR}(T(a_i), a_i) = \phi_i^{NR}(N, v_g), \tag{22}$$

and

$$\phi_i^{**}(N, v_g) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \Delta_{\bar{v}_g}(C_{\overrightarrow{\pi_i}}, a_i) = \frac{1}{|N|!} \sum_{T \in \Pi(N)} \Delta_{\bar{v}_g}(C_{\overleftarrow{\pi_i}}, a_i) = \phi_i^{SB}(N, v_g). \tag{23}$$

Since the correctness of Equation (23) is implied by Equation (19), it remains to prove the correctness of Equation (22). We will use the following lemma:

LEMMA 4.1.4. *Given (N, v_g) and $a_i \in N$, there exists a bijection $f_i : \Pi(N) \to \Pi(N)$ such that for all $\pi \in N$:*

$$\Delta_{v_{g,inv(\pi)}}(C_{\overrightarrow{\pi_i}}, a_i) = \Delta_{v_g}^{NR}(\underbrace{f_i(\pi)(a_i)}_{=T(a_i)}, a_i). \tag{24}$$

Proof. Let f_i be defined as follows: for every $\pi = (a_{k_1}, \ldots, a_{k_{i-1}}, a_{k_i}, a_{k_{i+1}}, \ldots, a_{k_n}) \in \Pi(N)$, where $a_{k_i} = a_i$, we have $f_i(\pi) = inv(\pi) = (a_{k_n}, \ldots, a_{k_{i+1}}, a_{k_i}, a_{k_{i-1}}, \ldots, a_{k_1})$. For this bijection, the right hand side of Equation (24) becomes:

$$\Delta_{v_g}^{NR}(f_i(\pi)(a_i), a_i) = \Delta_{v_g}^{NR}(inv(\pi)(a_i), a_i) = v_g((a_{k_n}, \ldots, a_{k_{i-1}}, a_{k_i})) - v_g((a_{k_n}, \ldots, a_{k_{i-1}})).$$

As for the left-hand side of Equation (24), we have:

$$\begin{aligned} \Delta_{v_{g,inv(\pi)}}(C_{\overrightarrow{\pi_i}}, a_i) &= v_{g,inv(\pi)}(C_{\overrightarrow{\pi_i}} \cup \{a_i\}) - v_{g,inv(\pi)}(C_{\overrightarrow{\pi_i}}) \\ &= v_{g,inv(\pi)}(\{a_{k_i}, a_{k_{i+1}}, \ldots, a_n\}) - v_{g,inv(\pi)}(\{a_{k_{i+1}}, \ldots, a_{k_n}\}). \end{aligned} \tag{25}$$

Since, by definition, we have $v_{g,\pi}(C) = v_g(\pi(C))$ for all $C \subseteq N$, we can rewrite Equation (25) as follows:

$$\begin{aligned} \Delta_{v_{g,inv(\pi)}}(C_{\overrightarrow{\pi_i}}, a_i) &= v_g(inv(\pi)(\{a_{k_i}, a_{k_{i+1}}, \ldots, a_n\}) - v_g(inv(\pi)(\{a_{k_{i+1}}, \ldots, a_{k_n}\})) \\ &= v_g((a_{k_n}, \ldots, a_{k_{i-1}}, a_{k_i})) - v_g((a_{k_n}, \ldots, a_{k_{i-1}})). \end{aligned} \tag{26}$$

Hence, Equation (24) holds. □

This concludes the proof of Theorem 1. □ □

Finally, we note that, based on Lemma 4.2.1, the equilibrium strategy from Dasgupta and Chiu [13] can be straightforwardly adapted to our $ODCG^{NR}$ and $ODCG^{SB}$ mechanisms (by replacing $v(C)$ with $v_g(inv(\pi(C \cup \{a_i\}))$ and $\frac{1}{|C|!}\sum_{T\in\Pi(C\cup\{a_i\})} v_g(T)$, respectively). The resulting equilibrium strategies are detailed in Appendix A.

5 Conclusions

Generalized characteristic function games are attracting increasing interest in the literature due to their manifold potential applications. In this article we the implementational aspects of these games. In particular, building upon the mechanism by Dasgupta and Chiu, we proposed the first mechanisms that implement the Nowak-Radzik value and the Sánchez-Bergantiños value.

Acknowledgments

Tomasz Michalak was supported by the European Research Council under Advanced Grant 291528 (“RACE”).

References

[1] G. Chalkiadakis, E. Elkind, and M. Wooldridge. *Computational Aspects of Cooperative Game Theory*. Synthesis Lectures on Artificial Intelligence and Machine Learning. Morgan & Claypool Publishers, 2011.

[2] A. Nowak and T. Radzik. The Shapley value for n-person games in generalized characteristic function form. *Games and Economic Behavior*, 6(1):150–161, 1994.

[3] A. Rutherford, M. Cebrian, I. Rahwan, S. Dsouza, J. McInerney, V. Naroditskiy, M. Venanzi, N. R Jennings, E. Wahlstedt, S. U Miller, et al. Targeted social mobilization in a global manhunt. *PLOS One*, 8(9):e74628, 2013.

[4] I. Rahwan, S. Dsouza, A. Rutherford, V. Naroditskiy, J. McInerney, M. Venanzi, N. R Jennings, and M. Cebrian. Global manhunt pushes the limits of social mobilization. *IEEE Computer*, 46(4):68–75, 2013.

[5] E. Sánchez and G. Bergantinños. Coalitional values and generalized characteristic functions. *Mathematical Methods of Operations Research*, 49(3):413–433, 1999.

[6] L. S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *In Contributions to the Theory of Games, volume II*, pages 307–317. 1953.

[7] E. Sánchez and G. Bergantiños. On values for generalized characteristic functions. *OR Spektrum*, 19(3):229–234, 1997.

[8] M. del Pozo, C. Manuel, E. González-Arangüena, and G. Owen. Centrality in directed social networks. a game theoretic approach. *Social Networks*, 33(3):191–200, 2011.

[9] R. Amer, José M. Giménez, and A. Magãna. A ranking for the nodes of directed graphs based on game theory. *Proceedings in Applied Mathematics and Mechanics*, 7(1), 2007.

[10] R. Amer, J. M. Giménez Pradales, and A. Magaña Nieto. Nodes of directed graphs ranked by solutions defined on cooperative games. In *XXXII Congreso Nacional de Estadística e Investigación Operativa"*, pages 1–20, 2010.

[11] S. J. Brams, H. Mutlu, and S. L. Ramirez. Influence in terrorist networks: From undirected to directed graphs. In *Studies in Conflict and Terrorism 29(7)*, 2006.

[12] H. K. Lentz, T. Selhorst, and I. M. Sokolov. Spread of infectious diseases in directed and modular metapopulation networks. *Physical Review E*, 85:066111, Jun 2012.

[13] A. Dasgupta and Y. S. Chiu. On implementation via demand commitment games. *International Journal of Game Theory*, 27(2):161–189, 1998.

[14] F. Gul. Bargaining foundations of Shapley value. *Econometrica*, 57(1):81–95, 1989.

[15] S. Hart and Z. Levy. Efficiency does not imply immediate agreement. *Econometrica*, 67(4):909–912, 1999.

[16] R. A. Evans. Value, consistency, and random coalition formation. *Games and Economic Behavior*, 12(1):68–80, 1996.

[17] D. Pérez-Castrillo and D. Wettstein. Bidding for the surplus: A non-cooperative approach to the Shapley value. *Journal of Economic Theory*, 100(2):274–294, 2001.

[18] D. Pérez-Castrillo and D. Wettstein. Implementation of the ordinal Shapley value for a three–agent economy. *Economics Bulletin*, 3(48):1–8, 2005.

[19] J. Vidal-Puga and G. Bergantiños. An implementation of the owen value. *Games and Economic Behavior*, 44(2):412–427, 2003.

[20] Y. Ju and D. Wettstein. Bidding with coalitional externalities: A strategic approach to partition function form games. *European Meeting of the Econometric Society in Budapest, Hungary*, pages 1–17, 2007.

[21] K. Hang P. Do and H. Norde. The Shapley value for partition function form games. *International Game Theory Review*, 09(02):353–360, 2007.

[22] I. Macho-Stadler, D. Pérez-Castrillo, and D. Wettstein. Efficient bidding with externalities. *Games and Economic Behavior*, 57(2):304–320, 2006.

[23] I. Macho-Stadler, D. Pérez-Castrillo, and D. Wettstein. Dividends and weighted values in games with externalities. *International Journal of Game Theory*, 39(1):177–184, March 2010.

[24] R. van den Brink and Y. Funaki. Axiomatization and implementation of discounted Shapley values. Tinbergen institute discussion papers, Tinbergen Institute, July 2010.

[25] G. Bergantinños and E. Sánchez. Weighted Shapley values for games in generalized characteristic function form. *TOP: An Official Journal of the Spanish Society of Statistics and Operations Research*, 9(1):55–67, 2001.

[26] R. Serrano. Fifty Years of the Nash Program, 1953-2003. Technical Report 2004-20, Brown University, 2004.

[27] Y. Shoham and K. Leyton-Brown. *Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations*. Cambridge University Press, 2009.

Appendix A: Strategies in the $ODCG^{NR}(v_g)$ and $ODCG^{SB}(v_g)$ mechanisms

In this appendix we present the equilibrium strategy of player a_i. It has two version; one for $ODCG^{NR}(v_g)$, denoted by $\sigma^{NR}_{\pi,i}$, and the other for$ODCG^{SB}(v_g)$, denoted by $\sigma^{SB}_{\pi,i}$. Before we introduce these strategies, we need to introduce additional notation. Let $H_i = \mathbb{R}^{i-1}$ be the set of all possible histories that a_i may face; every history in H_i represents a unique set of demands $(d_1, \ldots, d_{i-1})$. Now, let a_i face the history $h = (d_1, \ldots, d_{i-1})$ and let D be any subset of players from $\{a_i, a_{i+1}, \ldots, a_n\}$. We define $M^{NR}_{h,\pi}(D)$ as follows:

$$M^{NR}_{h,\pi}(D) = \max_{C \subseteq \{a_1,\ldots,a_{i-1}\}} \left\{ v_g(inv(\pi(D \cup C))) - \sum_{a_j \in C} d_j \right\}. \tag{27}$$

That is, $M^{NR}_{h,\pi}(D)$ is the maximum payoff that coalition D can obtain for itself if it is allowed to choose a set of new members, denoted by C, from the players that precede a_i, bearing in mind that the mechanism will enforce the formation of $inv(\pi(D \cup C))$. Furthermore, for every $j \in \{i, \ldots, n\}$, let $\mathcal{P}^h_\pi(i,j)$ denote the following linear program:

$$\begin{aligned}
\mathcal{P}^h_\pi(i,j)\ :\ \max_{d_i,\ldots,d_j}\ d_i \quad & \text{subject to:}\\
& d_{k_1} + \ldots + d_{k_m} \geqslant M^{NR}_{h,\pi}(\{a_{k_1}, \ldots, a_{k_m}\})\\
& \text{for all } k_1, \ldots, k_m \text{ where } i \leqslant k_1 < \ldots < k_m \leqslant j\\
& \text{and } d_i + \ldots + d_j = M^{NR}_{h,\pi}(\{a_i, \ldots, a_j\}).
\end{aligned}$$

In other words, $\mathcal{P}^h_\pi(i,j)$ computes for players $a_i, \ldots, a_j$ and characteristic function $M^{NR}_{h,\pi}(\cdot)$ a core allocation that gives a_i the largest payoff. Next, we outline the conditions that characterize the *maximal* program for player a_i:

Definition 6. *Given a history $h = (d_1, \ldots, d_{i-1})$, a program $\mathcal{P}^h_\pi(i,j)$ for player a_i is called a* ***maximal program*** *if:*

- *there exists a solution $(d_i, \ldots, d_j)$ for $\mathcal{P}^h_\pi(i,j)$, i.e. the program is feasible.*[3]
- *no other feasible program $\mathcal{P}^h_\pi(i,k) : k \neq j$ has a greater objective-function value;*
- *no other feasible program $\mathcal{P}^h_\pi(i,k) : k > j$ has the same objective-function value.*

The above definition implies that every maximal program is unique. We are now ready to introduce our strategy. Recall that every $a_i : 1 \leqslant i < n$ must choose between two options, either to specify a demand, or select a subset of $\{a_1, \ldots, a_{i-1}\}$, while a_n has only one option, which is to select a subset from $\{a_1, \ldots, a_{n-1}\}$). This implies that a strategy of $a_i : 1 \leqslant i < n$ is a mapping from H_i to $\mathbb{R} \cup 2^{\{a_1,\ldots,a_{i-1}\}}$, while a strategy of a_n is a mapping from H_n to $2^{\{a_1,\ldots,a_{n-1}\}}$. Our strategy, $\sigma^{NR}_{\pi,i}$, proceeds as follows:

[3]We note that the trivial program $\mathcal{P}^h_\pi(i,i)$ is always feasible.

- If $\mathcal{P}^h_\pi(i,j)$ is maximal, where $j > i$, then demand the value of the objective function in $\mathcal{P}^h_\pi(i,j)$
- If $\mathcal{P}^h_\pi(i,i)$ is maximal, then form the ordered coalition $inv(\pi(C^* \cup \{a_i\}))$, where C^* solves:

$$M^{NR}_{h,\pi}(\{a_i\}) = \max_{C \subseteq \{a_1,\ldots,a_{i-1}\}} \left\{ v_g(inv(\pi(C \cup \{a_i\}))) - \sum_{a_j \in C} d_j \right\}. \tag{28}$$

If there are more than one such argmaxes, then following Dasgupta and Chiu [13] we adopt the lexicographic tie-breaking rule.

As for the strategy $\sigma^{SB}_{\pi,i}$, it is identical to $\sigma^{NR}_{\pi,i}$ except for the following difference. Every $M^{NR}_{h,\pi}(D)$ is replaced with $M^{SB}_{h,\pi}(D)$, which is defined as follows:

$$M^{SB}_{h,\pi}(D) = \max_{C \subseteq \{a_1,\ldots,a_{i-1}\}} \left\{ \frac{1}{|C|!} \sum_{T \in \Pi(C \cup \{a_i\})} v_g(T) - \sum_{a_j \in C} d_j \right\}. \tag{29}$$

Appendix B: Main Notation Used in the Article

	Players, coalitions, permutations
N	The set of players.
n	Cardinality of set N.
a_i	A player in N.
N_{-i}	The set of players N without player a_i.
C, D	A coalition.
S, T	An ordered coalition.
$(T, S)^k$	The ordered coalition that results from inserting S at the k^{th} position in T.
(T, a_i)	The ordered coalition that results from inserting a_i at the end of T.
$\vec{\subseteq}$	An extension of the notion of a subset to ordered sets (see Definition 1).
$\Pi(C)$	The set of all possible permutations of the players in C.
π	A permutation.
$inv(\pi)$	The inverse of π.
$\pi(C)$	The ordered coalition which consists of all the players in C and which is ordered according to π.
	Value functions, games, solution concepts
v	The characteristic function.
v_g	The generalized characteristic function.
$v_{g,\pi}$	The characteristic function, where $v_{g,\pi}(C) = v_g(\pi(C))$.
(N, v)	A coalitional game in a characteristic function form.
(N, v_g)	A coalitional game in a generalized characteristic function form.
$(N, \bar{v}_g)$	The average game for v_g ($\bar{v}_g$ has the characteristic function form).
$C_{\overleftarrow{\pi}(i)}$	A coalition that consists of all the players that are in permutation π before a_i.
$\Delta_v(C_{\overleftarrow{\pi}(i)}, a_i)$	The marginal contribution of player a_i to C in the Shapley value (Equation 5).
$\Delta_{v_g}^{NR}(T, a_i)$	The marginal contribution of player a_i to T in the NR value.
$\Delta_{v_g}^{SB}(T, a_i)$	The marginal contribution of player a_i to T in the SB value (Equation 10).
$\Delta_{\bar{v}_g}(C_{\overleftarrow{\pi}(i)}, a_i)$	The marginal contribution of player a_i to C in the SB value (Equation 5).
$\phi_i(N, v)$	The Shapely value of player a_i in game (N, v).
$\phi_i^{NR}(N, v_g)$	The Radzik-Nowak value of player a_i in game (N, v_g).
$\phi_i^{SB}(N, v_g)$	The Sánchez-Bergantiños value of player a_i in game (N, v_g).
$\mathbb{E}[\cdot]$	The expectation operator.
	$ODCG^{NR}$ and $ODCG^{SB}$ mechanisms
d_i	A a "demand" made by player a_i.
$ODCG_\pi^{NR/SB}$	The subgame for the NR value and the SB value, respectively, given π.
H_i	The set of all histories that a_i can face.
h_i	A history in H_i.
$\sigma_{\pi,i}^{NR/SB}$	The strategy of player a_i in $ODCG_\pi^{NR/SB}$.
$M_{h,\pi}^{NR}(D)$	The maximum payoff that coalition D can obtain for itself if it is allowed to optimally choose a set of new members C from the players that precede a_i.
$\mathcal{P}_\pi^h(i, j)$	A linear program.